# Autoregressive model of 1/f noise

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*Physics Letters A Vol.257, no.1/2, p.37-42 (1999)*

(Received 7 August 1998; received in revised form 15 February 1999; accepted 28 April 1999)

## Abstract

An analytically solvable model is proposed exhibiting 1/f spectrum in any desirably wide range of frequency (but excluding the point $f = 0$). The model consists of pulses whose recurrence times obey an autoregressive process with very small damping.
PACS: 05.40.+j, 02.50.-r, 72.70.+m

Keywords: 1/f noise, Brownian motion, time series, long-memory processes, fluctuation phenomena

## 1. Introduction

The puzzle of the origin and omnipresence of 1/f noise – also known as 'flicker' or 'pink' noise – is one of the oldest unsolved problem of the contemporary physics. Since the first observation of flicker noise in the currents of electron tubes more than 70 years ago by Johnson [1] , fluctuations of signals and physical variables exhibiting behavior characterized by a power spectral density $S(f)$ diverging at low frequencies like $1/f^{\delta}$ ($\delta \simeq 1$) have been discovered in large diversity of uncorrelated systems. We can mention here processes in condensed matter, traffic flow, quasar emissions and music, biological, evolution and artificial systems and even human cognition (see [2–5] and references herein).

1/f noise is an intermediate between the well understood white noise with no correlation in time and the random walk (Brownian motion) noise with no correlation between increments. The widespread occurrence of signals exhibiting power spectral density with 1/f behavior suggests that a general mathematical explanation of such an effect might exist. However, except for some formal mathematical descriptions like "fractional Brownian motion" or half-integral of a white noise signal [6] no generally recognized physical explanation of the ubiquity of 1/f noise is still proposed. Models of 1/f noise in some physical systems are usually specialized (see [2–5] and references herein) and they do not explain the omnipresence of the processes with $1/f^{\delta}$ spectrum [7–9].

Note also some mathematical algorithms and models of the generation of the processes with 1/f noise [10–12]. These models can not, as a rule, be solved analytically and they do not reveal the origin as well as the necessary and sufficient conditions for the appearance of 1/f type fluctuations.

History of the progress in different areas of physics points to the crucial influence of simple models on the understanding of new phenomena. We note here only the decisive influence of the Lorenz model as well as the logistic and standard (Chirikov) maps for understanding of the deterministic chaos and the quantum kicked rotor for revealing the quantum localization of classical chaos.

It is the purpose of this letter to present and analyze the simplest analytically solvable model of 1/f noise which can be relevant for the understanding of the origin, main properties and parameter dependencies of flicker noise. Our model is a result of the search for necessary and sufficient conditions for the appearance of 1/f fluctuations in simple systems affected by random external perturbations, originated from the observation of a transition from chaotic to nonchaotic behavior in an ensemble of randomly driven systems [13], initiated in Ref. [14] and further developed in Ref. [15].

In the model there are analyzed currents or signals represented as sequences of random but correlated pulses whose recurrence times (intervals between transit times of pulses) obey an autoregressive process with small damping. It is shown that for small average recurrence time and very small damping, random increments of the recurrence times lead to 1/f behavior of the power spectrum of the signal or current in wide range of frequency, however, analytical at $f = 0$.

## 2. Model and solution

Let us consider a point process when the intensity of some signal consisting from a sequence of pulses (elementary events) or current of particles through some Poincaré section $L_m$ may be expressed as

$$I(t) = \sum_k a\delta(t - t_k). \tag{1}$$

Here $\delta(t)$ is the Dirac delta function, $\{t_k\}$ is a sequence of transit times $t_k$ at which the particles or pulses cross the section $L_m$ and $a$ is a contribution to the signal or current of one pulse or particle when it crosses the section $L_m$.

The power spectral density of the current (1) is

$$S(f) = \lim_{T\to\infty}\left\langle \frac{2a^2}{T}\left|\sum_{k=k_{\min}}^{k_{\max}} e^{-i2\pi f t_k}\right|^2\right\rangle = \lim_{T\to\infty}\left\langle \frac{2a^2}{T}\sum_k \sum_{q=k_{\min}-k}^{k_{\max}-k} e^{i2\pi f\Delta(k;q)}\right\rangle \tag{2}$$

where $\Delta(k;q) \equiv t_{k+q} - t_k$ is the difference of transit times $t_{k+q}$ and $t_k$, $T$ is the whole observation time interval, $k_{\min}$ and $k_{\max}$ are minimal and maximal values of index $k$ in the interval of observation and the brackets $\langle ... \rangle$ denote the averaging over realizations of the process.

Let us analyze a process whose recurrence times $\tau_k = t_k - t_{k-1}$ follow an autoregressive AR(1) process with offset $\bar{\tau} > 0$, regression coefficient $\alpha = 1 - \gamma$ and noise variance $\sigma^2$. So, defining by $\theta_k = \tau_k - \bar{\tau}$ a deviation of the recurrence time $\tau_k$ from the steady state value $\bar{\tau}$, we have autoregressive equations for the deviations $\theta_k$

$$\theta_k = \alpha\theta_{k-1} + \sigma\varepsilon_k \tag{3$a$}$$

and for the recurrence times $\tau_k$

$$\tau_k = \tau_{k-1} - \gamma(\tau_{k-1} - \bar{\tau}) + \sigma\varepsilon_k. \tag{3$b$}$$

Here $\{\varepsilon_k\}$ denotes a sequence of uncorrelated normally distributed random variables with zero expectation and unit variance (the white noise source) and $\sigma$ is the standard deviation of the white noise. Note that the coefficient $\gamma$ has a sense of damping (the relaxation rate of the recurrence times $\tau_k$ to the average value $\bar{\tau}$) and we will consider only processes with $\gamma \ll 1$ or $\gamma = 0$.

The recurrence equation for the transit times $t_k$ is

$$t_k = t_{k-1} + \tau_k \tag{4}$$

where the recurrence times $\tau_k$ is defined by Eq. (3b).

The simplest interpretation of our model corresponds to one particle moving along some orbit. The period of this motion fluctuates (due to external random perturbations of the system's parameters) about some average value $\bar{\tau}$. Some generalizations and extensions of the model and its interpretation will be discussed below.

From Eqs. (3a) and (3b) follows explicit expressions for the deviation $\theta_k$ and the recurrence time $\tau_k$,

$$\theta_k = \theta_0\alpha^k + \sigma\sum_{j=1}^{k}\alpha^{k-j}\varepsilon_j, \tag{5$a$}$$

$$\tau_k = \bar{\tau} + (\tau_0 - \bar{\tau})\alpha^k + \sigma\sum_{j=1}^{k}\alpha^{k-j}\varepsilon_j, \tag{5$b$}$$

with $\theta_0$ and $\tau_0$ being the initial values of the deviation and of the recurrence time, respectively.

The variance of the recurrence time $\tau_k$ is

$$\sigma_\tau^2(k) \equiv \left\langle \tau_k^2 \right\rangle - \langle \tau_k \rangle^2 = \sigma^2\left(1 - \alpha^{2k}\right)/\left(1 - \alpha^2\right) \tag{6}$$

After some algebra we can easily obtain from Eqs. (4) and (5b) an explicit expression for the transit times $t_k$,

$$t_k = t_0 + \sum_{j=1}^{k} \tau_j = t_0 + k\bar{\tau} + \frac{\alpha}{\gamma}\left(\tau_0 - \bar{\tau}\right)\left(1 - \alpha^k\right) + \frac{\sigma}{\gamma}\sum_{l=1}^{k}\left(1 - \alpha^{k+1-l}\right)\varepsilon_l, \tag{7}$$

with $t_0$ being the initial time.

The power spectral density of the current according to Eq. (2) depends on the statistics of the transit times difference $\Delta(k;q)$ which according to Eq. (7) is

$$\Delta(k;q) \equiv t_{k+q} - t_k = q\bar{\tau} + \frac{1}{\gamma}\left(\tau_0 - \bar{\tau}\right)\left(1 - \alpha^q\right)\alpha^{k+1}$$

$$+\frac{\sigma}{\gamma}\left[\left(1 - \alpha^q\right)\sum_{l=1}^{k}\alpha^{k+1-l}\varepsilon_l + \sum_{l=k+1}^{k+q}\left(1 - \alpha^{k+q+1-l}\right)\varepsilon_l\right], \quad q \geq 0. \tag{8}$$

Random variable $\Delta(k;q)$ is a sum of two regular terms and $k+q$ uncorrelated Gaussian random variables with zero expectations and variances $\left(\frac{\sigma}{\gamma}\right)^2\left(1-\alpha^q\right)^2\alpha^{2(k+1-l)}$ for $l = 1, 2, \ldots, k$ and $\left(\frac{\sigma}{\gamma}\right)^2\left(1-\alpha^{k+q+1-l}\right)^2$ for $l = k+1, k+2, \ldots, k+q$, respectively. Therefore, $\Delta(k;q)$ is a normally distributed random variable with the expectation

$$\mu_\Delta(k;q) \equiv \langle\Delta(k;q)\rangle = q\bar{\tau} + \frac{1}{\gamma}\left(\tau_0 - \bar{\tau}\right)\left(1 - \alpha^q\right)\alpha^{k+1} \tag{9}$$

and the variance $\sigma_\Delta^2(k;q) \equiv \left\langle\Delta(k;q)^2\right\rangle - \langle\Delta(k;q)\rangle^2$, which equals the sum of the variances of the components,

$$\sigma_\Delta^2(k;q) = \left(\frac{\sigma}{\gamma}\right)^2\left[\left(1-\alpha^q\right)^2\sum_{j=1}^{k}\alpha^{2j} + \sum_{j=1}^{q}\left(1-\alpha^j\right)^2\right]$$

$$= \left(\frac{\sigma}{\gamma}\right)^2\left[q - \frac{2\alpha\left(1-\alpha^q\right)}{1-\alpha^2} - \frac{\alpha^{2k+2}\left(1-\alpha^q\right)^2}{1-\alpha^2}\right], \quad q \geq 0. \tag{10}$$

Here new summation indexes $j = k+1-l$ and $j = k+q+1-l$ of first and second sums in (8), respectively, have been introduced.

Note that from the definition of the difference of transit times it follows the symmetry relations

$$\Delta(k;-q) = -\Delta(k-q;q), \quad \mu(k;-q) = -\mu(k-q;q), \quad \sigma_\Delta^2(k;-q) = \sigma_\Delta^2(k-q;q). \tag{11}$$

At $k \gg \gamma^{-1}$ or after averaging over $\tau_0$ from the distribution with the expectation $\bar{\tau}$ and variance $\sigma_\tau^2 \equiv \sigma_\tau^2(\infty) = \sigma^2/\left(1-\alpha^2\right) \simeq \sigma^2/2\gamma$ according to Eq. (6), expressions (7) and (8) generate a stationary time series. The expectation and the variance of the difference $\Delta(k;q)$ of transit times of the stationary time series are

$$\mu_\Delta(q) \equiv \langle\mu_\Delta(\infty;q)\rangle = q\bar{\tau}, \tag{12}$$

$$\sigma_\Delta^2(q) \equiv \sigma_\Delta^2(\infty;q) = \left(\frac{\sigma}{\gamma}\right)^2\left[q - \frac{2\alpha\left(1-\alpha^q\right)}{1-\alpha^2}\right]. \tag{13}$$

The power spectral density of the current according to Eq. (2) is

$$S(f) = \lim_{T\to\infty} \frac{2a^2}{T} \sum_{k,q} \left\langle e^{i2\pi f \Delta(k;q)} \right\rangle = \lim_{T\to\infty} \frac{2a^2}{T} \sum_{k,q} \chi_{\Delta(k;q)}(2\pi f) \tag{14}$$

where $\chi_{\Delta(k;q)}(2\pi f)$ is the characteristic function of the distribution of the transit times difference $\Delta(k;q)$. For the normal distribution of $\Delta(k;q)$ the characteristic function takes the form $\chi_{\Delta(k;q)}(2\pi f) = \exp\left[i2\pi f \mu_\Delta(k;q) - 2\pi^2 f^2 \sigma_\Delta^2(k;q)\right]$ and the power spectral density equals

$$S(f) = \lim_{T\to\infty} \frac{2a^2}{T} \sum_{k,q} \exp\left[i2\pi f \mu_\Delta(k;q) - 2\pi^2 f^2 \sigma_\Delta^2(k;q)\right]. \tag{15}$$

For $q \ll \gamma^{-1}$ we have from Eqs. (9) and (10) expansions of the expectation $\mu_\Delta(k;q)$ and of the variance $\sigma_\Delta^2(k;q)$ in powers of $\gamma q \ll 1$

$$\mu_\Delta(k;q) = \tau(k)\, q, \quad \tau(k) = \bar{\tau} + (\tau_0 - \bar{\tau})\, \alpha^{k+1}, \tag{16}$$

$$\sigma_\Delta^2(k;q) = \frac{\sigma^2}{2} \left\{ \left[ \frac{2\left(1-\alpha^{2k}\right)}{1-\alpha^2} + \alpha^{2k} \right] q^2 + \left( \alpha^{2k} - \frac{1}{3} \right) q^3 + \frac{1}{3} q \right\}. \tag{17}$$

The leading term of the expansion (17) may be written as

$$\sigma_\Delta^2(k;q) = \sigma_\tau^2(k)\, q^2, \quad \gamma q \ll 1 \tag{18}$$

where the variance $\sigma_\tau^2(k)$ of the recurrence time $\tau_k$ is defined by Eq. (6). Therefore, Eq. (15) takes the form

$$S(f) = \lim_{T\to\infty} \frac{2a^2}{T} \sum_{k,q} \exp\left[i2\pi f \tau(k)\, q - 2\pi^2 f^2 \sigma_\tau^2(k)\, q^2\right]. \tag{19}$$

Here the mean recurrence time $\tau(k)$ is defined in Eq. (16).

Eq. (19) is valid if $2\pi^2 f^2 \sigma_\tau^2(k)\, q^2 \mid_{q=\gamma^{-1}} \gg 1$, i.e., for $f > f_1 = \gamma/\pi\sigma_\tau(k)$. When $f \ll f_\tau = [2\pi\tau(k)]^{-1}$ and $f < f_2 = [\pi\sigma_\tau(k)]^{-1}$ we can replace the summation over $q$ in Eq. (19) by the integration. The integration yields to the 1/f spectrum

$$S(f) = \frac{a^2}{f} \sqrt{\frac{2}{\pi}} \lim_{T\to\infty} \frac{1}{T} \sum_k \frac{1}{\sigma_\tau(k)} \exp\left[ -\frac{\tau(k)^2}{2\sigma_\tau^2(k)} \right], \quad f_1 < f < \min\{f_2, f_\tau\}. \tag{20}$$

Eq. (20) is valid for the stationary as well as for the non-stationary process with slowly changing of the expectation and variance of the recurrence time.

At $k \gg \gamma^{-1}$ or after averaging over $\tau_0$ from the distribution with the expectation $\bar{\tau}$ and variance $\sigma_\tau^2 = \sigma^2/(1-\alpha^2) \simeq \sigma^2/2\gamma$ we have the stationary process: the expectation and the variance of the recurrence time do not depend on the parameter $k$, i.e., $\tau(k) = \bar{\tau}$ and $\sigma_\tau(k) = \sigma_\tau$. For the stationary process Eq. (20) takes the form

$$S(f) = \bar{I}^2 \frac{\alpha_H}{f}. \tag{21}$$

Here $\bar{I} = \lim_{T\to\infty} a(k_{\max} - k_{\min} + 1)/T = a/\bar{\tau}$ is the average current and $\alpha_H$ is a dimensionless constant (the Hooge parameter)

$$\alpha_H = \frac{2}{\sqrt{\pi}} K e^{-K^2}, \quad K = \frac{\bar{\tau}}{\sqrt{2}\sigma_\tau}. \tag{22}$$

Therefore, the power of 1/f noise except of the squared average current strongly depends on the ratio of the average recurrence time to the standard deviation of the recurrence time.

**3. Discussion and generalizations**

The point process containing only one relaxation time $\gamma^{-1}$ can for sufficiently small damping $\gamma$ and average recurrence time $\bar{\tau} \ll \sigma/\sqrt{\gamma}$ (with $\sigma$ being the standard deviation of the white noise source) produce an exact 1/f-like spectrum in wide range of frequency $(f_1, f_2)$, with $f_2/f_1 \simeq \gamma^{-1}$. Furthermore, due to the contribution to the transit times $t_k$ of the large number of the random variables $\varepsilon_l$ $(l = 1, 2, ...k)$, our model represents a 'long-memory' random process. As a result of the nonzero relaxation rate $(\gamma \neq 0)$ and, consequently, due to the finite variance, $\sigma_\tau^2 = \sigma^2/2\gamma$, of the recurrence time the model is free from the unphysical divergency of the spectrum at $f \to 0$. So, using an expansion of expression (13) at $\gamma q \gg 1$ , $\sigma_\Delta(q) = (\sigma/\gamma)^2 q$, we obtain from Eq. (15) the spectrum density $S(f) = \bar{I}^2 (2\sigma^2/\bar{\tau}\gamma^2)$ for $f \ll \min\{f_1, f_0 = \bar{\tau}\gamma^2/\pi\sigma^2\}$. This is in agreement with the statement [16] that the power spectrum of any pulse sequence is white at low enough frequencies.

This simple, consistent and exactly solvable model can easily be generalized in different directions: for large number of particles moving in similar orbits with coherent (identical for all particles) or independent (uncorrelated for different particles) fluctuations of the periods, for non-Gaussian or continuous perturbations of the systems' parameters, for nonlinear relaxation and for spatially extended systems. So, when an ensemble of $N$ particles moves on closed orbits and the period of each particle fluctuates independently (due to the perturbations by uncorrelated sequences of random variables $\{\varepsilon_k^v\}$, different for each particle $v$) the power spectral density of the collective current $I$ of all particles can be calculated by the above method too and is expressed as the Hooge formula [2]

$$S(f) = \bar{I}^2 \frac{\alpha_H}{N f}. \tag{23}$$

The model may be used for evaluation of the power spectral density of the non-stationary process as well. So, at $k \ll \gamma^{-1}$ or for $\gamma = 0$ we have a process with the constant averaged recurrence time $\tau(k) = \tau_0$, from Eq. (16), and linearly increasing variance of the recurrence time $\sigma_\tau^2(k) = \sigma^2 k$, according to Eq. (6), i.e., a process similar to the Brownian motion without relaxation. For $k \gg |q|$ it is valid expansion (18) and the power spectral density of such process for finite observation time interval $T$ may be evaluated according to Eq. (20)

$$S(f) = \frac{1}{f}\sqrt{\frac{2}{\pi}}\frac{a^2}{\sigma T}\sum_{k=k_{\min}}^{k_{\max}}\frac{1}{\sqrt{k}}\exp\left[-\frac{\tau_0^2}{2\sigma^2 k}\right], \quad f_1 \ll f < f_2. \tag{24}$$

Here

$$f_1 = \left(\sigma k_{\max}\sqrt{k_{\min}}\right)^{-1}, \quad f_2 = \left(\pi\sigma\sqrt{k_{\max}}\right)^{-1}. \tag{25}$$

The process with random increments of the recurrence time and without relaxation is, however, very unstable and strongly depending on the realization. Really, the averaged number, $\langle k_{\max} - k_{\min}\rangle$, of transition times $t_k$ for the given observation time interval $T$ is $\langle k_{\max} - k_{\min}\rangle = T/\tau_0$ but the standard deviation $\sigma_T$ of the time interval for given $k_{\min}$ and $k_{\max}$ according to Eq. (10) equals $\sigma_T = \sigma(k_{\max} - k_{\min})\sqrt{(k_{\max} + 2k_{\min})/3}$. Therefore, for $(k_{\max} + 2k_{\min}) > 3(\tau_0/\sigma)^2$ the standard deviation exceeds the expectation value $\langle T\rangle = (k_{\max} - k_{\min})\tau_0$ of the time interval. The power spectrum of any realization is, however, of 1/f type for large frequency interval $(f_1, f_2)$, with $f_2/f_1 \simeq \sqrt{k_{\max}k_{\min}}$.

It should be noticed in conclusion that in many cases the intensity of signals or currents can be expressed in the form (1). This expression represent exactly the flow of identical point objects. More generally, in Eq. (1) instead of the Dirac delta function one should introduce time dependent pulse amplitudes $A_k(t - t_k)$. However, the low frequency power spectral density depends weakly on the shapes of the pulses [16], while fluctuations of the pulses amplitudes result, as a rule, in white or Lorentzian, but not 1/f, noise. The model (2)–(5) in such cases represents fluctuations of the averaged period $\tau_k$ between the subsequent transition times of the pulses. Therefore, the model may be easily generalized and applied for the explanation of 1/f noise in different systems. Furthermore, it reveals the possible origin of 1/f noise, i.e. random increments of the time intervals between the pulses or elementary events.

Summarizing, a simple analytically solvable model of 1/f noise is presented and analyzed. The model reveals main features and parameter dependences of the power spectral density of the noise. The model and its generalizations may essentially influence the understanding of the origin and main properties of the flicker noise.

**Acknowledgements**

The author acknowledges stimulating discussions with Prof. R. Katilius, Prof. A. Matulionis, Dr. A. Bastys and Dr. T. Meškauskas. The research of this publication was made possible in part by support of the Alexander von Humboldt Foundation and Lithuanian State Science and Studies Foundation.

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